INDEPENDENCE COMPLEXES OF CLAW-FREE GRAPHS

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Abstract. We study the class of independence complexes of claw-free graphs. The main theorem give good bounds on the connectivity of these complexes, given bounds for a few subcomplexes of the same class. Two applications are presented. Firstly, we show that the independence complex of a claw-free graph with n vertices and maximal degree d is $(cn/d + \varepsilon)$–connected, where $c = 2/3$. This can be compared with the result of Szabó and Tardos that $c = 1/2$ is optimal with no restrictions on the graphs. Secondly, we calculate the connectivity of a family of complexes used in Babson and Kozlov's proof of Lovász conjecture.

1. Introduction

The independence complex is a good structure for transferring graph coloring problems to combinatorial topology. Usually the topological statements to investigate will be about connectivity. In this paper we study the connectivity of independence complexes of claw-free graphs.

First let us fix notation and introduce some tools.

1.1. **Graphs.** All graphs are finite and simple. For a graph G the edge set is $E(G)$ and the vertex set $V(G)$. A complete graph has edges between all vertices. The complement of G is called $\overline{G}$. The induced subgraph of G on $U \subseteq V(G)$ is denoted $G[U]$, and $G \setminus U = G[V(G) \setminus U]$. A set $I \subseteq V(G)$ is independent if $G[I]$ lacks edges. The set of vertices of a graph G with edges to a vertex v, is the neighborhood of v. It is called $N_G(v)$, or just $N(v)$. And $\dot{N}(v) = N(v) \cup \{v\}$.

1.2. **Topological tools.** All topological tools used are standard. For proofs and further references see Björner's survey [2] chapters 9–10. A topological space T is n-connected if for all $0 \leq i \leq n$ any map from the i–sphere to T can be extended to a map from the $(i+1)$–ball to T. Arcwise connected and 0–connected is the same. Define all non-empty spaces to be (-1)-connected, and all spaces to be n–connected for $n \leq -2$. These lemmas will be used several times:

Lemma 1.1 (Corollary of Theorem 10.6 [2],Theorem 1.1 [3])**.** *If $\Delta_1, \Delta_2, \ldots \Delta_k$ are n–connected simplicial complexes and $\cap_{i \in I} \Delta_i$ is $(n-1)$–connected for any $\emptyset \neq I \subseteq \{1, 2, \ldots k\}$ then $\cup_{i=1}^k \Delta_i$ is n–connected.*

Lemma 1.2 (Theorem 10.4 [2])**.** *If $\Delta_0, \Delta_1, \ldots \Delta_k$ are contractible simplicial complexes and $\Delta_i \cap \Delta_j \subseteq \Delta_0$ for all $1 \leq i < j \leq k$ then $\cup_{i=0}^k \Delta_i = \vee_{i=1}^k \mathrm{susp}(\Delta_0 \cap \Delta_i)$.*

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If Δ is a simplicial complex with vertex set V and $U \subseteq V$, then the induced subcomplex is $\Delta[U] = \{\sigma \in \Delta \mid \sigma \subseteq U\}$.

2. Independence complexes of claw-free graphs

2.1. **Claw-free graphs.** A *claw* is four vertices u, v_1, v_2, v_3 with edges from u to v_1, v_2, v_3, but no edges among v_1, v_2, v_3. A graph is *claw-free* if there are no induced subgraphs which are claws. An equivalent definition is:

Definition 2.1. A graph G is *claw-free* if $\overline{G[N(u)]}$ is triangle-free for all $u \in V(G)$.

Lemma 2.2. *If u is a vertex of a claw-free graph G, and $v \in N(u)$, then $G[N(v) \setminus \dot{N}(u)]$ is a complete graph.*

Proof. Let w_1, w_2 be two arbitrary vertices of $G[N(v) \setminus \dot{N}(u)]$. There are edges from v to w_1, w_2 and u, and no edges from u to w_1 and w_2. An edge between w_1 and w_2 is the only way to avoid a claw. □

2.2. **Independence complexes.**

Definition 2.3. Let G be a graph. The *independence complex of G*, $\text{Ind}(G)$ has vertex set $V(G)$ and its simplices are the independent subsets of $V(G)$.

Some basic properties are:

* If $U \subseteq V(G)$, then $\text{Ind}(G)[U] = \text{Ind}(G[U])$.
* If $u \in V(G)$ then $\text{Ind}(G \setminus N(u))$ is a cone with apex u.
* If $u \in V(G)$ and $\sigma \in \text{Ind}(G)$ then there is a $v \in \dot{N}(u)$ such that $\sigma \cup \{v\} \in \text{Ind}(G)$.
* If $u, v \in V(G)$ and $\{u, v\}$ is a connected component of G, then $\text{Ind}(G) \simeq \text{susp}(\text{Ind}(G \setminus \{u, v\}))$.

Two results from [4] are needed. The proofs are short, so they are included for completeness.

Lemma 2.4. *If $N(v) \subseteq N(w)$ then $\text{Ind}(G)$ collapses onto $\text{Ind}(G \setminus \{w\})$.*

Proof. Let $\{\sigma_1, \sigma_2, \ldots \sigma_k\} = \{\sigma \in \text{Ind}(G) \mid w \in \sigma, v \notin \sigma\}$ be ordered such that if $\sigma_i \supseteq \sigma_j$ then $i < j$. The successive removals from $\text{Ind}(G)$ of $\{\sigma_1, \sigma_1 \cup \{v\}\}$, $\{\sigma_2, \sigma_2 \cup \{v\}\}, \ldots, \{\sigma_k, \sigma_k \cup \{v\}\}$ are elementary collapse steps. □

Lemma 2.5. *If $u \in V(G)$ and $G[N(u)]$ is a complete graph, then*

$$\text{Ind}(G) \simeq \bigvee_{v \,\in\, N(u)} \text{susp}(\text{Ind}(G \setminus \dot{N}(v)))$$

Proof. Let $\Delta_v = \text{Ind}(G \setminus N(v))$ for all $v \in \dot{N}(u)$. All Δ_v are contractible, and $\Delta_{v_1} \cap \Delta_{v_2} \subseteq \Delta_u$ for all distinct $v_1, v_2 \in N(u)$. By Lemma 1.2, and the third basic property of independence complexes listed above,

$$\text{Ind}(G) = \bigcup_{v \,\in\, \dot{N}(u)} \text{Ind}(G \setminus N(v)) = \bigcup_{v \,\in\, \dot{N}(u)} \Delta_v \simeq \bigvee_{v \,\in\, N(u)} \text{susp}(\Delta_u \cap \Delta_v) = \bigvee_{v \,\in\, N(u)} \text{susp}(\text{Ind}(G \setminus \dot{N}(v)))$$

□

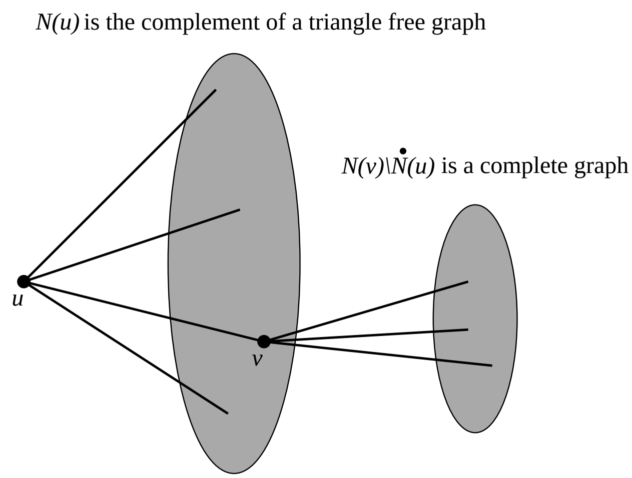

FIGURE 1. The local structure of a claw-free graph

2.3. **Higher connectivity.** Lemma 2.5 is a good tool for calculating the homotopy type of independence complexes of graphs where neighborhoods which form complete subgraphs can be found. In general this is not the case for claw-free graphs, but as illustrated in Figure 1, the situation is quite similar. It is probably impossible to use the local structure of claw-free graphs to calculate the homotopy type of their independence complexes recursively without running into devastating identifications on the resulting topological space. However, in Theorem 2.8 we show that the connectivity of independence complexes of claw-free graphs can be handled.

Lemma 2.6. *If* $u, v \in V(G)$, $N(u) = \{v\}$, *and* $\mathrm{Ind}(G \setminus \dot{N}(v))$ *is* $(n-1)$*–connected, then* $\mathrm{Ind}(G)$ *is* n*–connected.*

Proof. The neighborhood of every vertex in $N(v) \setminus \{u\}$ contains v, and v is the only vertex adjacent to u. Hence $\mathrm{Ind}(G)$ collapses onto $\mathrm{Ind}(G \setminus (N(v) \setminus \{u\}))$ by repeated use of Lemma 2.4. The vertices u and v form a connected component of $G \setminus (N(v) \setminus \{u\})$, so $\mathrm{Ind}(G \setminus (N(v) \setminus \{u\})) \simeq \mathrm{susp}(\mathrm{Ind}((G \setminus (N(v) \setminus \{u\})) \setminus \{u, v\})) = \mathrm{susp}(\mathrm{Ind}(G \setminus \dot{N}(v)))$. Since $\mathrm{Ind}(G \setminus \dot{N}(v))$ is $(n-1)$–connected, $\mathrm{Ind}(G)$ is n–connected. $\square$

Lemma 2.7. *Let* G *be a graph with three vertices* u, v_1, *and* v_2, *such that* $\{v_1, v_2\} \notin E(G)$, $N(u) = \{v_1, v_2\}$, *and both* $G[N(v_1) \setminus \{u\}]$ *and* $G[N(v_2) \setminus \{u\}]$ *are complete graphs. If* $\mathrm{Ind}(G \setminus (\dot{N}(u) \cup (N(v_1) \cap N(v_2))))$ *is* $(n-1)$*–connected and* $\mathrm{Ind}(G \setminus (\dot{N}(w_1) \cup \dot{N}(w_2) \cup \{u\}))$ *is* $(n-2)$*–connected for every* $\{w_1, w_2\} \in E(\overline{G[N(v_1) \cup N(v_2) \setminus \{u\}]})$, *then* $\mathrm{Ind}(G)$ *is* n*–connected.*

Proof. Let $H = G \setminus (N(v_1) \cap N(v_2))$. First we prove that $\mathrm{Ind}(H)$ is n–connected, and then the rest follows easily. If $N_G(v_1) \subseteq N_G(v_2)$ then v_1 is isolated in H and $\mathrm{Ind}(H)$ is a cone with apex v_1 and n–connected. Now assume that $N_G(v_1) \not\subseteq N_G(v_2)$. The vertices v_1 and v_2 of H have disjoint and complete neighborhoods, which fits good

with using Lemma 2.5 twice,

$$\mathrm{Ind}(H) \simeq \bigvee_{w_1 \,\in\, N_H(v_1)} \mathrm{susp}(\mathrm{Ind}(H \setminus \dot{N}_H(w_1)))$$

and

$$\mathrm{Ind}(H \setminus \dot{N}_H(w_1)) = \bigvee_{w_2 \,\in\, N_H(v_2) \setminus \dot{N}_H(w_1)} \mathrm{susp}(\mathrm{Ind}(H \setminus (\dot{N}_H(w_1) \cup \dot{N}_H(w_2)))).$$

There is an edge between w_1 and w_2 in $\overline{G[N_G(v_1) \cup N_G(v_2) \setminus \{u\}]}$ if and only if $w_1 \in N_H(v_1) = N_G(v_1)\setminus\{u\}$ and $w_2 \in N_H(v_2)\setminus\dot{N}_H(w_1) = (N_G(v_2)\setminus\{u\})\setminus\dot{N}_H(w_1)$. We assumed that $\mathrm{Ind}(H\setminus(\dot{N}_H(w_1)\cup\dot{N}_H(w_2))) = \mathrm{Ind}(G\setminus(\dot{N}_G(w_1)\cup\dot{N}_G(w_2)\cup\{u\}))$ is $(n-2)$–connected for every $\{w_1, w_2\} \in E(\overline{G[N(v_1) \cup N(v_2) \setminus \{u\}]})$, therefore $\mathrm{Ind}(H \setminus \dot{N}_H(w_1))$ is $(n-1)$–connected for every $w_1 \in N_H(v_1)$. Well, actually not for all $w_1 \in N_H(v_1)$ because of that. If $\dot{N}_H(w_1) \supset N_H(v_2)$, then we cannot use Lemma 2.5 a second time, but then $\mathrm{Ind}(H \setminus \dot{N}_H(w_1))$ is a cone with apex v_2 and $(n-1)$–connected.

All $\mathrm{Ind}(H \setminus \dot{N}_H(w_1))$ are $(n-1)$–connected, so $\mathrm{Ind}(H) = \mathrm{Ind}(G \setminus (N_G(v_1) \cap N_G(v_2)))$ is n-connected. The intersection of $\mathrm{Ind}(G\setminus(N_G(v_1)\cap N_G(v_2)))$ and $\mathrm{Ind}(G\setminus N_G(u))$ is $\mathrm{Ind}(G \setminus (\dot{N}_G(u) \cup (N_G(v_1) \cap N_G(v_2))))$ which is assumed to be $(n-1)$-connected. $\mathrm{Ind}(G \setminus N_G(u))$ is a cone with apex u and n–connected. Thus the union of $\mathrm{Ind}(G\setminus(N_G(v_1)\cap N_G(v_2)))$ and $\mathrm{Ind}(G\setminus N_G(u))$, $\mathrm{Ind}(G\setminus(N_G(v_1)\cap N_G(v_2)\setminus\{u\}))$, is n–connected. Finally, by repeated use of Lemma 2.4, $\mathrm{Ind}(G)$ collapses onto $\mathrm{Ind}(G \setminus (N_G(v_1) \cap N_G(v_2) \setminus \{u\}))$ since $N_G(w) \supset N_G(u)$ for all $w \in N_G(v_1) \cap N_G(v_2) \setminus \{u\}$, and hence $\mathrm{Ind}(G)$ is n–connected. □

Theorem 2.8. *Let u be a vertex of a claw-free graph G. If*

- $*$ $\mathrm{Ind}(G\setminus\dot{N}(v))$ *is* $(n-1)$*–connected for every* $v \in N(u)$ *such that* $\dot{N}(v) \supseteq \dot{N}(u)$,
- $*$ $\mathrm{Ind}(G \setminus (\dot{N}(u) \cup (N(v_1) \cap N(v_2))))$ *is* $(n-1)$*–connected for every* $\{v_1, v_2\} \in E(\overline{G[N(u)]})$,
- $*$ $\mathrm{Ind}(G \setminus (\dot{N}(u) \cup \dot{N}(w_1) \cup \dot{N}(w_2)))$ *is* $(n-2)$*–connected for every* $\{w_1, w_2\} \in E(\overline{G[N(v_1) \cup N(v_2) \setminus \dot{N}(u)]})$ *where* $\{v_1, v_2\} \in E(\overline{G[N(u)]})$,

then $\mathrm{Ind}(G)$ *is n–connected.*

Proof. Define $\Delta_v = \mathrm{Ind}(G \setminus (N(u) \setminus \{v\}))$ for all $v \in N(u)$, and $\Delta_{v_1,v_2} = \mathrm{Ind}(G \setminus (N(u) \setminus \{v_1, v_2\}))$ for all $\{v_1, v_2\} \in E(\overline{G[N(u)]})$.

Any face of $\mathrm{Ind}(G)$ either contains a vertex from $\dot{N}(u)$ or can be extend with it. There is a face of $\mathrm{Ind}(G)$ with two distinct vertices v_1, v_2 of $\dot{N}(u)$ exactly when $\{v_1, v_2\} \in E(\overline{G[N(u)]})$. But there can never be three vertices since the complement of a neighborhood in a claw-free graph is triangle-free. A vertex v of $N(u)$ such that $\dot{N}(v) \supseteq \dot{N}(u)$ can never be together with another vertex from $N(u)$ in a face of $\mathrm{Ind}(G)$. We can cover $\mathrm{Ind}(G)$:

$$\mathrm{Ind}(G) \quad = \bigcup_{\substack{v\in N(u)\\ \dot{N}(v)\supseteq\dot{N}(u)}} \Delta_v \quad \cup \bigcup_{\{v_1,\, v_2\} \,\in\, E(\overline{G[N(u)]})} \Delta_{v_1,v_2}$$

We will now show that the subcomplexes we cover with are n–connected and that their intersections are $(n-1)$–connected. From that we can conclude that $\mathrm{Ind}(G)$ is n–connected by Lemma 1.1. The cases are:

(a) Δ_v is n–connected for all $v \in N(u)$ such that $\dot{N}(v) \supseteq \dot{N}(u)$.

(b) Δ_{v_1,v_2} is n–connected for all $\{v_1, v_2\} \in E(\overline{G[N(u)]})$.
(c) The intersection of at least two different subcomplexes from (a) and (b) is $(n-1)$–connected:
  (i) One of the subcomplexes is a Δ_v.
  (ii) None of the subcomplexes is a Δ_v, and there are two subcomplexes Δ_{v_1,v_2} and Δ_{v_3,v_4} such that $\{v_1, v_2\} \cap \{v_3, v_4\} = \emptyset$.
  (iii) The subcomplexes are $\Delta_{v,v_1}, \Delta_{v,v_2}, \dots \Delta_{v,v_k}$.

Case a. Let v be a vertex of $N(u)$ such that $\dot{N}(v) \supseteq \dot{N}(u)$. The neighborhood of u in $G \setminus (N(u) \setminus \{v\})$ is $\{v\}$, so by Lemma 2.6, $\Delta_v = \mathrm{Ind}(G \setminus (N(u) \setminus \{v\}))$ is n–connected since

$$\mathrm{Ind}((G \setminus (N(u) \setminus \{v\})) \setminus \dot{N}_{G\setminus(N(u)\setminus\{v\})}(v)) = \mathrm{Ind}(G \setminus \dot{N}(v))$$

is $(n-1)$–connected by assumption.

Case b. Let $\{v_1, v_2\}$ be an edge of $\overline{G[N(u)]}$ and define $H = G\setminus(N(u)\setminus\{v_1, v_2\})$. We are to prove that $\Delta_{v_1,v_2} = \mathrm{Ind}(H)$ is n-connected, and we will us Lemma 2.7 to do that. Let's check the conditions of the lemma. The three vertices we use are u, v_1, v_2.

* $\{v_1, v_2\} \notin E(H)$.
* $N_H(u) = \{v_1, v_2\}$.
* By Lemma 2.2, $H[N_H(v_1) \setminus \{u\}] = G[N_G(v_1) \setminus \dot{N}_G(u)]$ is a complete graph.
* By the same reason $H[N_H(v_2) \setminus \{u\}]$ is a complete graph.
* From the inclusions $H \subset G$ and $N_G(u) \setminus \{v_1, v_2\} \subseteq \dot{N}_G(u) \cup (N_G(v_1) \cap N_G(v_2)))$ we get that $\mathrm{Ind}(H \setminus (\dot{N}_H(u) \cup (N_H(v_1) \cap N_H(v_2)))) = \mathrm{Ind}(G \setminus (\dot{N}_G(u) \cup (N_G(v_1) \cap N_G(v_2))))$ which is $(n-1)$–connected by assumption.
* In the same way $\mathrm{Ind}(H \setminus (\dot{N}_H(w_1) \cup \dot{N}_H(w_2) \cup \{u\}))$ is $(n-2)$–connected for every $\{w_1, w_2\} \in E(\overline{H[N_H(v_1) \cup N_H(v_2) \setminus \{u\}]})$, since $\mathrm{Ind}(G\setminus(\dot{N}_G(u) \cup \dot{N}_G(w_1) \cup \dot{N}_G(w_2)))$ is $(n-2)$–connected for every $\{w_1, w_2\} \in E(\overline{G[N_G(v_1) \cup N_G(v_2) \setminus \dot{N}_G(u)]})$ where $\{v_1, v_2\} \in E(\overline{G[N_G(u)]})$ by assumption.

Case c. First note that the intersection with any of the subcomplexes Δ_v and Δ_{v_1,v_2} with $\mathrm{Ind}(G \setminus N(u))$ is $\mathrm{Ind}(G \setminus N(u))$. And that is a cone with apex u and thus contractible. After sufficient many intersections of subcomplexes we will see that one ends up with $\mathrm{Ind}(G \setminus N(u))$ for which the connectedness is allright.

Case c.i. Say that one of the subcomplexes is Δ_{v_1}. If $v_1 \neq v_2$ then $\Delta_{v_1} \cap \Delta_{v_2} = \mathrm{Ind}(G \setminus N(u))$. If $\{v_2, v_3\} \in E(\overline{G[N(u)]})$ and $\dot{N}(v_1) \supseteq \dot{N}(u)$ then $v_1 \notin \{v_2, v_3\}$ and $\Delta_{v_1} \cap \Delta_{v_2,v_3} = \mathrm{Ind}(G \setminus N(u))$. We conclude that a intersection where one of the subcomplexes is Δ_{v_1} is $(n-1)$–connected.

Case c.ii. The intersection of two subcomplexes Δ_{v_1,v_2} and Δ_{v_3,v_4} such that $\{v_1, v_2\} \cap \{v_3, v_4\} = \emptyset$ is $\mathrm{Ind}(G \setminus N(u))$ so the complete complete intersection is also $\mathrm{Ind}(G \setminus N(u))$ which is $(n-1)$–connected.

Case c.iii. $\cap_{i=1}^{k} \Delta_{v,v_i} = \mathrm{Ind}(G \setminus (N(u) \setminus v))$. We assumed that $\mathrm{Ind}(G \setminus (\dot{N}(v) \cup \dot{N}(u)))$ is $(n-2)$–connected for every $v \in N(u)$ such that $\dot{N}(v) \not\supseteq \dot{N}(u)$. By Lemma 2.6, $\mathrm{Ind}(G \setminus (N(u) \setminus v))$ is $(n-1)$–connected. □

3. Asymptotic higher connectivity

It was proved in [4, Theorem 26] that for any graph G with maximal degree d, $\mathrm{Ind}(G)$ is $(\lfloor (n - 2d - 1)/2d \rfloor)$–connected, where d is the maximal degree of a

vertex of G. For a graph property, it is an interesting task to find the best c, such that for G with the property, $\mathrm{Ind}(G)$ is $f(n,d)$–connected where $f(d,d)$ grows asymptotically as cn/d. In [4, 6] it was proved that $c = 1/2$ if we put no restriction on the graphs. In this section we prove that $c \geq 2/3$ for claw-free graphs.

Lemma 3.1. *If G is a claw-free graph with maximal degree d, $u \in V(G)$, and $\{v_1, v_2\} \subseteq N(u)$ but $\{v_1, v_2\} \notin E(G)$, then*

$$\#\dot{N}(u) \cup (N(v_1) \cap N(v_2)) \leq \lfloor (3d+2)/2 \rfloor$$

Proof. For every vertex in the neigborhood of u other than v_1 and v_2, at least one of v_1 and v_2 must have an edge to it since G is claw-free. Therefore either v_1 or v_2 must have edges to at least half of the elements of $N(u) \setminus \{v_1, v_2\}$. Assume that it is v_1. Insert

$$\#N(u) \cap N(v_1) \geq \lceil (\#N(u)-2)/2 \rceil \Rightarrow \#\dot{N}(u) \cap N(v_1) \geq \lceil \#N(u)/2 \rceil$$

into

$$\begin{array}{rcl} \#N(v_1) \cap N(v_2) \setminus \dot{N}(u) & \leq & \#N(v_1) \setminus \dot{N}(u) \\ & = & \#N(v_1) - \#N(v_1) \cap \dot{N}(u) \\ & \leq & \#N(v_1) - \lceil \#N(u)/2 \rceil \end{array}$$

to conclude that

$$\begin{array}{rcl} \#\dot{N}(u) \cup (N(v_1) \cap N(v_2)) & = & \#\dot{N}(u) + \#N(v_1) \cap N(v_2) \setminus \dot{N}(u) \\ & \leq & \#\dot{N}(u) + \#N(v_1) - \lceil \#N(u)/2 \rceil \\ & = & 1 + \#N(u) + \#N(v_1) - \lceil \#N(u)/2 \rceil \\ & = & 1 + \#N(v_1) + \lfloor \#N(u)/2 \rfloor \\ & \leq & 1 + d + \lfloor d/2 \rfloor \\ & = & \lfloor (3d+2)/2 \rfloor \end{array}$$

□

Theorem 3.2. *If G is a claw-free graph with n vertices and maximal degree d, then $\mathrm{Ind}(G)$ is $\lfloor (2n-1)/(3d+2) - 1 \rfloor$–connected.*

Proof. If $d = 0$ the statement is true, so assume that $d \geq 1$. If $0 < n \leq (3d+2)/2$ the statement is that $\mathrm{Ind}(G)$ is (-1)–connected. This means that the complex is nonempty, which is true. The proof is by induction over the number of vertices. Note that subgraphs of G never have higher maximal degree than d.

Assume that $n > (3d+2)/2$ and fix a vertex u of G. The independence complex of G is broken up into smaller pieces with bounded connectivity and patched together with Theorem 2.8. The next step is to check that the conditions of the theorem are fullfilled.

- Let v be a vertex in $N(u)$. There are at most $d+1$ elements in $\dot{N}(v)$, and $(3d+1)/2 \geq d+1$, so $\mathrm{Ind}(G \setminus \dot{N}(v))$ is $(\lfloor (2n-1)/(3d+2) - 1 \rfloor - 1)$–connected by induction.
- By Lemma 3.1 $\#\dot{N}(u) \cup (N(v_1) \cap N(v_2)) \leq \lfloor (3d+2)/2 \rfloor$ for every $\{v_1, v_2\}$ in $E(\overline{G[N(u)]})$. Thus $\mathrm{Ind}(G \setminus (\dot{N}(u) \cup (N(v_1) \cap N(v_2))))$ is $(\lfloor (2n-1)/(3d+2) - 1 \rfloor - 1)$–connected by induction.
- For every $\{w_1, w_2\} \in E(\overline{G[N(v_1) \cup N(v_2) \setminus \dot{N}(u)]})$ where $\{v_1, v_2\} \in E(\overline{G[N(u)]})$, the intersection of $\dot{N}(u)$ and $\dot{N}(w_1) \cup \dot{N}(w_2)$ contains v_1 and v_2, so $\#\dot{N}(u) \cup \dot{N}(w_1) \cup \dot{N}(w_2) \leq 3d+1$. Therefore $\mathrm{Ind}(G \setminus (\dot{N}(u) \cup \dot{N}(w_1) \cup \dot{N}(w_2)))$ is $(\lfloor (2n-1)/(3d+2) - 1 \rfloor - 2)$–connected by induction.

We conclude by Theorem 2.8 that $\mathrm{Ind}(G)$ is $\lfloor(2n-1)/(3d+2)-1\rfloor$–connected. □

4. Connectivity of $\mathcal{C}_n^k$

We will treat two classes of independence complexes of claw-free graphs introduced by Kozlov [5]. Let L_n^k be the graph with vertex set $\{1, 2, \ldots, n\}$ and two vertices $i < j$ are adjacent if $j - i < k$. Define $\mathcal{L}_n^k = \mathrm{Ind}(L_n^k)$. For $n \leq 0$ let $\mathcal{L}_n^k = \emptyset$. In Engström [4, Corollary 21] it was proved that

$$\mathcal{L}_n^k \simeq \bigvee_{1 \leq i < \min\{k,n\}} \mathrm{susp}(\mathcal{L}_{n-k-i}^k)$$

using something like Lemma 2.5. It follows directly that $\mathcal{L}_n^k$ is $l_{n,k}$–connected, where

$$l_{n,k} = \left\lfloor \frac{n-1}{2k-1} - 1 \right\rfloor.$$

The second class is build from C_n^k which is a graph with vertex set $\{1, 2, \ldots, n\}$ and two vertices $i < j$ are adjacent if $j - i < k$ or $(n + i) - j < k$. Define $\mathcal{C}_n^k = \mathrm{Ind}(C_n^k)$. The homotopy type of $\mathcal{C}_n^2$ was determined in [5], and used by Babson and Kozlov in their proof of Lovász conjecture [1]. Some other cases where treated in [4], but in general the homotopy type of $\mathcal{C}_n^k$ is not known. Removing at least k consecutive vertices from $\mathcal{C}_n^k$ gives a complex of the $\mathcal{L}$ type which we know the higher connectivity of. We will cover $\mathcal{C}_n^k$ with $\mathcal{L}$ type complexes and then use Theorem 2.8 to bound the connectivity of it. Why is C_n^k claw-free? If we for example pick three elements of $N(k)$, then two of them must be either larger or smaller than k, which forces their difference smaller than k, and they are adjacent.

Theorem 4.1. *If $n \geq 6(k-1)$ then $\mathcal{C}_n^k$ is $c_{n,k}$–connected, where*

$$c_{n,k} = \left\lfloor \frac{n+1}{2k-1} - 2 \right\rfloor.$$

Proof. We are to check the conditions of Theorem 2.8. Let $u = 3k - 2$.

* There no $v \in N(u)$ such that $\dot{N}(v) \subseteq \dot{N}(u)$.
* If $\{v_1, v_2\} \in E(\overline{G[N(u)]})$ then $N(v_1) \cap N(v_2) \subseteq \dot{N}(u)$, so $\mathrm{Ind}(G \setminus (\dot{N}(u) \cup (N(v_1) \cap N(v_2)))) = \mathrm{Ind}(G \setminus (\dot{N}(u))) \simeq \mathcal{L}_{n-(2k-1)}^k$ which is $l_{n-(2k-1),k}$–connected. Clearly $c_{n,k} - 1 \leq l_{n-(2k-1),k}$.
* Choose $v_1 = 2k-1, v_2 = 4k-3, w_1 = k$, and $w_2 = 5k-4$ to minimize the size of $\mathrm{Ind}(C_n^k \setminus (\dot{N}(u) \cup \dot{N}(w_1) \cup \dot{N}(w_2))) \simeq \mathcal{L}_{n-(6k-5)}^k$ which is $l_{n-(6k-5),k}$–connected. Clearly $c_{n,k} - 2 = l_{n-(6k-5),k}$.

□

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